

# RATIONALITY OF THE QUOTIENT OF $\mathbb{P}^2$ BY FINITE GROUP OF AUTOMORPHISMS OVER ARBITRARY FIELD OF CHARACTERISTIC ZERO

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ABSTRACT. Let  $k$  be a field,  $\text{char } k = 0$  and  $G$  be a finite group of automorphisms of  $\mathbb{P}_k^2$ . Castelnuovo's Theorem implies that the quotient variety  $\mathbb{P}_k^2/G$  is rational if the field  $k$  is algebraically closed. In this paper we prove that the quotient  $\mathbb{P}_k^2/G$  is rational for an arbitrary field  $k$  of characteristic zero.

## 1. INTRODUCTION

Let  $G$  be a finite group and  $k$  be a field. Consider a pure transcendental extension  $K/k$  of transcendental degree  $n = \text{ord } G$ . We may assume that  $K = k\{(x_g)\}$ , where  $g$  runs through all the elements of group  $G$ . The group  $G$  naturally acts on  $K$  as  $h(x_g) = x_{hg}$ . Noether's problem asks whether the field of invariants  $K^G$  is rational (i.e. pure transcendental) over  $k$  or not. On the language of algebraic geometry, this is a question about the rationality of the quotient variety  $\mathbb{A}^n/G$ .

The most complete answer to this question is known for abelian groups, but even in this case quotient variety can be non-rational (see [Swa69], [Vos-foi] [EM73], [Len74]).

Noether's problem can be generalized as follows. Let  $G$  be a finite group, let  $V$  be finite-dimensional vector space over an arbitrary field  $k$  and let  $\rho : G \rightarrow GL(V)$  be a representation. The question is if the quotient variety  $V/G$  is  $k$ -rational?

Note that  $V/G$  has a natural birational structure of a  $\mathbb{P}^1$ -fibration over  $\mathbb{P}(V)/G$ , which is locally trivial in Zarisky topology. So rationality of  $V/G$  follows from rationality of  $\mathbb{P}(V)/G$ .

In this generalization it is natural to start with a low-dimensional case.

The most general result is known for dimension 1 and 2.

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**Theorem 1.1** (Lüroth). *Let  $k$  be an arbitrary field and let  $G \subset PGL_2(k)$  be a finite subgroup. Then  $\mathbb{P}_k^1/G$  is  $k$ -rational.*

The next theorem is a consequence of Castelnuovo's Theorem [Cast].

**Theorem 1.2.** *Let  $k$  be an algebraically closed field of characteristic zero and let  $G \subset PGL_3(k)$  be a finite subgroup. Then  $\mathbb{P}_k^2/G$  is  $k$ -rational.*

The main result of this paper is the following.

**Theorem 1.3.** *Let  $k$  be an arbitrary field of characteristic zero and let  $G \subset PGL_3(k)$  be a finite subgroup. Then  $\mathbb{P}_k^2/G$  is  $k$ -rational.*

**Corollary 1.4.** *Let  $k$  be an arbitrary field of characteristic zero and let  $G \subset GL_3(k)$  be a finite subgroup. The field of invariants  $k(x_1, x_2, x_3)^G$  is  $k$ -rational.*

To prove this statement we consider algebraical closure of the field  $k$ . We have two groups acting on  $\mathbb{P}_k^2$ : the geometrical group  $G$  and the Galois group  $\Gamma = \text{Gal}(\bar{k}/k)$ . Then we consider the quotient variety  $\mathbb{P}_k^2/N$  where  $N$  is a normal subgroup of  $G$  (if such a subgroup  $N$  exists). Next, we resolve the singularities of  $\mathbb{P}_k^2/N$ , run the  $G/N \times \Gamma$ -equivariant minimal model program and get a surface  $X$ . Then we repeat the above procedure applying this method to the surface  $X$  and the group  $G/N$ .

In the section 2 we describe notions and results of minimal model program which are used in this work. In the section 3 we sketch the classification of finite subgroups in  $PGL_3(\bar{k})$  where  $\bar{k}$  is an algebraically closed field of characteristic zero. In the section 4 we prove Theorem 1.3.

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We use the following notation.

$k$  denotes an arbitrary field of characteristic zero.

$\bar{k}$  denotes the algebraic closure of a field  $k$ .

$C_n$  denotes the cyclic group of order  $n$ .

$D_{2n}$  denotes the dihedral group of order  $2n$ .

$\mathfrak{S}_n$  denotes the symmetric group of degree  $n$ .

$\mathfrak{A}_n$  denotes the alternating group of degree  $n$ .

$\omega = e^{\frac{2\pi i}{3}}$ .

$I_n$  denotes the identity matrix of dimension  $n$ .

$$\text{diag}(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

$K_X$  denotes the canonical divisor of a variety  $X$ .

$\text{Pic}(X)$  (resp.  $\text{Pic}(X)^G$ ) denotes the (invariant) Picard group of a variety  $X$ .

$\rho(X)$  (resp.  $\rho(X)^G$ ) denotes the (invariant) Picard number of a variety  $X$ .

$\mathbb{F}_n$  denotes the minimal rational ruled (Hirzebruch) surface  $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ .

$X \approx Y$  denotes birationally equivalence between varieties  $X$  and  $Y$ .

## 2. $G$ -EQUIVARIANT MINIMAL MODEL PROGRAM

In this section we follow papers [DI-fs], [DI-po], [Isk79].

**Definition 2.1.** A *rational surface*  $X$  is a smooth projective surface over  $k$  such that  $\overline{X} = X \otimes \overline{k}$  is birationally isomorphic to  $\mathbb{P}_{\overline{k}}^2$ .

**Definition 2.2.** A  $G$ -*surface* is a pair  $(X, \rho)$  where  $X$  is a smooth projective surface and  $\rho$  is a monomorphism  $G \hookrightarrow \text{Aut}(X)$ . A morphism of surfaces  $f : X \rightarrow X'$  is called a morphism of  $G$ -surfaces  $(X, \rho) \rightarrow (X', \rho')$  if  $\rho'(G) = f \circ \rho(G) \circ f^{-1}$ .

**Definition 2.3.** A  $G$ -surface  $(X, \rho)$  is called *minimal* if any birational morphism of  $G$ -surfaces  $(X, \rho) \rightarrow (X', \rho')$  is an isomorphism.

Note that in our case there are two groups acting on  $\overline{X}$ : the geometrical group  $G$  and the Galois group  $\Gamma = \text{Gal}(\overline{k}/k)$  and the action of  $G$  is  $\Gamma$ -equivariant. It means that for each  $g \in G$ ,  $\gamma \in \Gamma$  and  $x \in \overline{X}$  one has  $\gamma\rho(g)x = \rho(g)\gamma x$ . Throughout this paper by *minimal surface* we mean  $(G \times \Gamma)$ -minimal surface.

The classification of minimal rational surfaces is well-known due to S. Mori. We introduce some important notions before surveying it.

**Definition 2.4.** A rational  $G$ -surface  $(X, \rho)$  admits a structure of a *conic bundle* if there exists a  $G$ -equivariant morphism  $\phi : X \rightarrow \mathbb{P}^1$  such that any fibre is isomorphic to a reduced conic in  $\mathbb{P}^2$ .

Note that a general fibre of  $\phi$  is isomorphic to  $\mathbb{P}_{\overline{k}}^1$  and its self-intersection equals 0. At the same time there may be *singular fibres* each of which being a pair of intersecting  $(-1)$ -curves. It is clear that if a conic bundle is  $G$ -minimal then two components of each singular fibre are permuted by the group  $G$ .

We will use the next theorem to work with conic bundles.

**Theorem 2.5.** [Isk79, Theorem 4] *Let  $X \rightarrow \mathbb{P}^1$  be a conic bundle. Then  $X$  is not minimal if  $K_X^2 \in \{3, 5, 6, 7\}$ .*

**Definition 2.6.** A *Del Pezzo surface* is a smooth projective surface  $X$  such that the anticanonical divisor  $-K_X$  is ample.

**Theorem 2.7.** [Isk79, Theorem 1] *Let  $X$  be a minimal rational  $G$ -surface. Then either  $X$  admits a structure of conic bundle with  $\text{Pic}(X)^G \cong \mathbb{Z}^2$ , or  $X$  is isomorphic to a Del Pezzo surface with  $\text{Pic}(X)^G \cong \mathbb{Z}$ .*

The next theorem is an important criterium for proving  $k$ -rationality over an arbitrary perfect field  $k$  (see [Isk96]).

**Theorem 2.8.** *A minimal rational surface  $X$  over a perfect field  $k$  is  $k$ -rational if and only if the following two conditions are satisfied:*

- (i)  $X(k) \neq \emptyset$ ;
- (ii)  $d = K_X^2 \geq 5$ .

Note that all surfaces in this work are rational by Theorem 1.2. Taking a quotient, resolving singularities and running a minimal model program don't affect the existence of  $k$ -points, so there exists a  $k$ -point on each considered surface. Therefore in this work if  $X$  is a smooth surface and  $K_X^2 \geq 5$  then  $X$  is  $k$ -rational.

The important type of rational surfaces is toric surfaces.

**Definition 2.9.** *Toric variety* is a normal variety containing an algebraic torus as a dense subset.

*Remark 2.10.* A minimal rational surface  $X$  is toric if and only if  $K_X^2 \geq 6$ .

A minimal rational surface  $X$  with  $K_X^2 \geq 6$  is  $\mathbb{P}_k^2$ ,  $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ , del Pezzo surface of degree 6 or a minimal rational ruled surface  $\mathbb{F}_n$  ( $n \geq 2$ ).

### 3. FINITE SUBGROUPS IN $PGL_3(\mathbb{C})$

**Definition 3.1.** Any finite subgroup of  $GL_n(k)$  is called a *linear group* in  $n$  variables.

We will use detailed classification of finite linear subgroups in 3 variables over an algebraically closed field of characteristic zero.

Let a linear group  $G$  act on the space  $V = k^3$  where  $k$  is algebraically closed.

**Definition 3.2.** If the action of the group  $G$  on  $V$  is reducible the group  $G$  is called *intransitive*. Otherwise the group  $G$  is called *transitive*.

**Definition 3.3.** Let  $G$  be a transitive group. If there exists a decomposition  $V = V_1 \oplus \cdots \oplus V_l$  to subspaces such that for any element  $g \in G$  one has  $gV_i = V_j$  then the group  $G$  is called *imprimitive*. Otherwise the group  $G$  is called *primitive*.

**Lemma 3.4.** *Let  $k$  be an algebraically closed field of characteristic zero. Then any representation of a finite group  $G$  in  $GL_n(k)$  is conjugate to a representation of the group  $G$  in  $GL_n(\overline{\mathbb{Q}})$*

According Lemma 3.4 for our purpose it is sufficient to know the classification of finite subgroups of  $GL_3(\mathbb{C})$ . In the classification we do not distinguish between groups which are equivalent modulo scalar multiplications because they define the same subgroup in  $PGL_3(\mathbb{C})$ . Thus we need to know the classification of finite subgroups of  $SL_3(\mathbb{C})$  modulo scalar multiplications.

We use the following notation:  $S = \text{diag}(1, \omega, \omega^2)$ ,  $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ ,  $V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$ ,  $U = \text{diag}(\varepsilon, \varepsilon, \varepsilon\omega)$ ,  $\varepsilon^3 = \omega^2$ .

The finite subgroups of  $SL_3(\mathbb{C})$  were completely classified in [Bl17, Chapter V] and [MBD16, Chapter XII].

**Theorem 3.5.** *Any finite subgroup in  $SL_3(\mathbb{C})$  (modulo scalar multiplications) is conjugate to one of the following.*

*Intransitive group:*

- (A) *A diagonal abelian group.*
- (B) *A group having a unique invariant subspace of dimension 2.*

*Imprimitive group:*

- (C) *A group having a normal abelian subgroup  $N$  such that  $G/N \cong C_3$ .*
- (D) *A group having a normal abelian subgroup  $N$  such that  $G/N \cong \mathfrak{S}_3$ .*

*Primitive groups having normal subgroups:*

- (E) *Group of order 108 generated by  $S$ ,  $T$  and  $V$ .*
- (F) *Group of order 216 generated by (E) and  $P = UVU^{-1}$ .*
- (G) *Group of order 648 generated by (E) and  $U$ .*

*Simple groups:*

- (H) *Group of order 60 isomorphic to the group  $\mathfrak{A}_5$ .*
- (I) *Klein group of order 168 isomorphic to the group  $PSL_2(\mathbb{F}_7)$ .*
- (K) *Valentiner group  $G$  of order 1080 i.e., its quotient  $G/F$  is isomorphic to the group  $\mathfrak{A}_6$ , where  $F$  is the group generated by  $\omega I_3$ .*

#### 4. RATIONALITY OF THE QUOTIENT VARIETY

In this section for each finite group  $G \subset PGL_3(\mathbb{C})$  (see Theorem 3.5) we prove that the quotient variety  $\mathbb{P}_k^2/G$  is  $k$ -rational. The main method is the following. We consider the algebraic closure  $\bar{k}$  of the field  $k$ . Two groups act on  $\mathbb{P}_{\bar{k}}^2$ : the geometrical group  $G$  and the Galois group  $\Gamma = \text{Gal}(\bar{k}/k)$ . In addition the action of the group  $G$  is  $\Gamma$ -equivariant.

If the group  $G$  is cyclic or simple,  $k$ -rationality of  $\mathbb{P}_k^2/G$  is proved in the first and the last cases of this section.

Otherwise there is a normal subgroup  $N$  in the group  $G$ . We consider the quotient variety  $\mathbb{P}_{\bar{k}}^2/N$ , resolve singularities, run the  $G/N \times \Gamma$ -equivariant minimal model program and get a  $G/N \times \Gamma$ -minimal surface  $X$ . One has

$$\mathbb{P}_{\bar{k}}^2/G = (\mathbb{P}_{\bar{k}}^2/N)/(G/N) \approx X/(G/N),$$

therefore it is sufficient to prove that  $X/(G/N)$  is  $k$ -rational. If there is a normal group  $M$  in the group  $G/N$  we can repeat this method. We will use the following definition for convenience.

**Definition 4.1.** Let  $S$  be a  $G \times \Gamma$ -surface,  $\tilde{S} \rightarrow S$  be its minimal resolution of singularities and  $Y$  be a  $G \times \Gamma$ -equivariant minimal model of  $\tilde{S}$ . We denote the surface  $Y$  by  $G \times \Gamma$ -MMP-reduction of  $S$ .

For short we will write an *MMP-reduction* instead of a  $G \times \Gamma$ -MMP reduction.

**4.1. Diagonal abelian groups.** Each abelian subgroup  $G \subset SL_3(\bar{k})$  is conjugate to a diagonal subgroup, so its action on  $\mathbb{P}_{\bar{k}}^2$  can be considered as the action of finite subgroup in an open torus in  $\mathbb{P}_{\bar{k}}^2$ .

**Lemma 4.2.** *Let  $X$  be a toric variety over an algebraic closed field  $\bar{k}$ ,  $\text{char } \bar{k} = 0$  and let  $G$  be a finite subgroup of a torus. Then the quotient  $X/G$  is a toric variety.*

*Proof.* Let  $\mathbb{T}^n$  be an open torus in  $X$ . The regular function's algebra of  $\mathbb{T}^n$  is  $\bar{k}[x_1, \dots, x_n, \frac{1}{x_1}, \dots, \frac{1}{x_n}]$  and its monoms form a lattice  $\mathbb{Z}^n$ . The action of the group  $G$  on this algebra is monomial so monoms of the algebra of  $G$ -invariants form a sublattice in this lattice. It means that  $\mathbb{T}^n/G$  is a torus in  $X/G$ , so  $X/G$  is a toric variety.  $\square$

*Remark 4.3.* Note that the resolution of singularities and minimal model programm don't affect toric structure on a surface. So for a toric surface  $X$  and finite subgroup  $G$  of a torus one has an MMP-reduction of  $X/G$  is a minimal toric surface. Therefore if there exists a  $k$ -point on  $X/G$  then it is  $k$ -rational by Theorem 2.8.

Let  $G$  be a finite abelian subgroup in  $PGL_3(\bar{k})$ . Then the MMP-reduction of  $\mathbb{P}_{\bar{k}}^2/G$  is a minimal toric surface by Lemma 4.2, it is  $k$ -rational by Theorem 2.8.

**4.2. Groups having a unique fixed point.** In this case the group  $G$  acts on  $\bar{k}^3$  and there exists decomposition  $\bar{k}^3 = \bar{k}^2 \oplus \bar{k}$  into  $G$ -invariant linear spaces. Moreover, the one-dimensional subspace  $\bar{k}$  is  $\Gamma$ -invariant because the action of the group  $G$  is  $\Gamma$ -equivariant and there are no other one-dimensional  $G$ -invariant subspaces. Therefore there exists decomposition  $k^3 = k^2 \oplus k$  into  $G$ -invariant linear subspaces.

It means that there is a unique  $G$ -fixed point  $p \in \mathbb{P}_k^2$  and a unique  $G$ -invariant line  $l$ . Let  $\mathbb{F}_1$  be the blowup of  $\mathbb{P}_k^2$  at the point  $p$ . The surface  $\mathbb{F}_1$  admits a  $G$ -equivariant  $\mathbb{P}_k^1$ -bundle structure  $\mathbb{F}_1 \rightarrow \mathbb{P}_k^1$  which fibres are proper transforms of lines passing through the point  $p$ . Obviously this  $\mathbb{P}_k^1$ -bundle has  $G$ -invariant sections: the exceptional divisor of the blowup at  $p$  and the proper transform of  $l$ . So one has  $\mathbb{F}_1/G \approx \mathbb{P}_k^1 \times \mathbb{P}_k^1/G$ . Therefore our problem is reduced to a one-dimensional case.

**4.3. Imprimitive groups.** Each imprimitive group  $G$  contains a normal abelian subgroup  $N$  conjugate to a diagonal abelian subgroup in  $GL_3(\bar{k})$ . The quotient group  $G/N$  is isomorphic to  $C_3$  or  $\mathfrak{S}_3$  (it corresponds to cases (C) and (D) of Theorem 3.5). Moreover a MMP-reduction of  $\mathbb{P}_{\bar{k}}^2/N$  is a  $k$ -rational minimal toric surface by Lemma 4.2. In this subsection we prove the following proposition:

**Proposition 4.4.** *Let  $X$  be a  $k$ -rational minimal toric surface and let  $G$  be a group  $C_3$  or  $\mathfrak{S}_3$   $\Gamma$ -equivariantly acting on  $X$ . Then  $X/G$  is  $k$ -rational.*

In the proof of this proposition quotient singularities of definite types play important role. The following remark is useful to work with them.

*Remark 4.5.* Let a group  $G$  act on a surface  $X$  and fix a point  $p \in X$ . Let  $f : X \rightarrow X/G$  is the quotient map. Then one has:

For the action of the group  $G = C_2$  on a tangent space at the point  $p$  as  $-I_2$  the point  $f(p)$  is a du Val singularity of type  $A_1$ ,

For the action of the group  $G = C_3$  on a tangent space at the point  $p$  as  $\text{diag}(\omega, \omega^2)$  the point  $f(p)$  is a du Val singularity of type  $A_2$ .

Moreover these singularities have the following properties:

(a) The minimal resolution  $\pi : Y \rightarrow X/C_2$  of a singularity  $A_1$  gives the exceptional divisor which is a  $(-2)$ -curve; one has  $K_Y^2 = K_{X/G}^2$ .

For nonsingular curve  $C_{X/C_2}$  passing through the singularity and  $C_Y = \pi^*C_{X/C_2}$  we have  $C_Y^2 = C_{X/C_2}^2 - \frac{1}{2}$ .

(b) The minimal resolution  $\pi : Y \rightarrow X/C_3$  of a singularity  $A_2$  gives the exceptional divisor which consists of two components. Each of them is a  $(-2)$ -curve and they intersect transversally at one point; one has  $K_Y^2 = K_{X/C_3}^2$ .

Let  $C_X$  be a  $C_3$ -invariant nonsingular irreducible curve on  $X$  passing through the point  $p$ ,  $C_{X/C_3} = f(C_X)$  and  $C_Y = \pi^*C_{X/C_3}$ . Then  $C_Y^2 = C_{X/C_3}^2 - \frac{2}{3}$ .

Let  $C_X$  and  $D_X$  be two  $C_3$ -invariant nonsingular irreducible curves on  $X$  which intersect transversally at the point  $p$ . Then the curves  $C_Y = \pi^*f(C_X)$  and  $D_Y = \pi^*f(D_X)$  intersect with different components of the exceptional divisor.

Now we come to the proof of Proposition 4.4.

**Lemma 4.6.** *Let  $X$  be a  $k$ -rational minimal toric surface,  $p$  be prime and the group  $C_p$   $\Gamma$ -equivariantly act on  $X$ . Then an MMP-reduction of  $X/C_p$  is a  $k$ -rational minimal toric surface.*

The Proposition 4.4 follows from this Lemma. If the group  $G = C_3$  it directly follows from Lemma 4.6. If the group  $G = \mathfrak{S}_3$  then a surface  $Y$ , which is  $C_2 \times \Gamma$ -MMP reduction of  $X/C_3$ , is a  $k$ -rational minimal toric surface by Lemma 4.6 and MMP-reduction of  $Y/C_2$  is a  $k$ -rational minimal toric surface by Lemma 4.6.

To prove Lemma 4.6 we case-by-case consider  $\mathbb{P}_{\bar{k}}^2$ ,  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ , del Pezzo surface of degree 6 and minimal rational ruled surfaces  $\mathbb{F}_n$  ( $n \geq 2$ ) with an action of the group  $C_p$ .

4.3.1. *Case 1:*  $\mathbb{P}_{\bar{k}}^2$ . Each cyclic group  $C_n$  is a finite subgroup of an open torus in  $\mathbb{P}_{\bar{k}}^2$ . Therefore an MMP-reduction of  $\mathbb{P}_{\bar{k}}^2/C_n$  is a  $k$ -rational minimal toric surface by Lemma 4.2.

4.3.2. *Case 2:*  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ . The automorphism group of  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$  is isomorphic to the group  $(PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2$ . We have the exact sequence:

$$1 \rightarrow PGL_2(\bar{k}) \times PGL_2(\bar{k}) \hookrightarrow (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2 \twoheadrightarrow C_2 \rightarrow 1.$$

Let  $p$  be a prime and  $C_p$  be a cyclic subgroup of the group  $\text{Aut}(\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1) = (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2$ . The composition of maps

$$C_p \hookrightarrow (PGL_2(\bar{k}) \times PGL_2(\bar{k})) \rtimes C_2 \twoheadrightarrow C_2$$

takes  $C_p$  to identity if  $p \neq 2$ . Thus one has  $C_p \hookrightarrow PGL_2(\bar{k}) \times PGL_2(\bar{k})$ . So the group  $C_p$  is a finite subgroup of an open torus in  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ . An MMP-reduction of  $(\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1)/C_p$  is a  $k$ -rational minimal toric surface by Lemma 4.2.



If  $p = 2$  and  $C_2$  is not a subgroup of  $PGL_2(\bar{k}) \times PGL_2(\bar{k})$  then the action of the group  $C_2$  is conjugate to

$$(x_1 : x_0; y_1 : y_0) \mapsto (y_1 : y_0; x_1 : x_0)$$

where  $(x_1 : x_0; y_1 : y_0)$  are homogeneous coordinates on  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ . There is a fixed curve  $\frac{x_1}{x_0} = \frac{y_1}{y_0}$  on  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$  which class in  $\text{Pic}(\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1)$  equals  $-\frac{1}{2}K_{\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1}$ . The quotient variety  $Y = (\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1)/C_2$  is nonsingular. By Hurvitz formula one has  $K_Y^2 = \frac{1}{2}(\frac{3}{2}K_X)^2 = 9$ . Thus the surface  $Y$  is isomorphic to  $\mathbb{P}_{\bar{k}}^2$  and toric.

**4.3.3. Case 3: Minimal ruled surfaces  $\mathbb{F}_n$  ( $n \geq 2$ ).** Let the group  $G$  act on  $\mathbb{F}_n$ . Then the conic bundle structure  $\mathbb{F}_n \rightarrow \mathbb{P}_{\bar{k}}^1$  is  $G$  equivariant. It means that there exists the exact sequence:

$$1 \rightarrow G_F \hookrightarrow G \rightarrow G_B \rightarrow 1$$

where  $G_F$  is a group of automorphisms of general fibre and  $G_B$  is a group of automorphisms of the base  $B = \mathbb{P}_{\bar{k}}^1$ . Therefore for a prime  $p$  the action of the group  $C_p$  on the base is either faithful or trivial.

In the first case there are two fixed points on the base  $\mathbb{P}_{\bar{k}}^1$ . The corresponding fibres of  $\mathbb{F}_n \rightarrow \mathbb{P}_{\bar{k}}^1$  are  $C_p$ -invariant. In the second case all fibres of  $\mathbb{F}_n \rightarrow \mathbb{P}_{\bar{k}}^1$  are  $C_p$ -invariant. So in the both cases we can choose  $C_p$ -invariant fibre  $F_1$ .

The action of the group  $C_p$  on  $F_1$  is either faithful or trivial. Therefore there are at least two  $C_p$ -fixed points. So we can choose a fixed point  $p$  which don't lay on the  $(-n)$ -section. Let us blow up the point  $p$  and contract the proper transform of  $F_1$ . We get a surface  $\mathbb{F}_{n-1}$  with the action of the group  $C_p$ . By repeating of this procedure  $n$  times we can obtain a  $C_p$ -equivariant birational map  $f : \mathbb{F}_n \dashrightarrow \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ .

Note that  $\rho(\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1)^{C_p} = \rho(\mathbb{F}_n)^{C_p} = 2$ . Therefore if  $p = 2$  the group  $C_2$  can't act as a permutation of the rulings of  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ . So the group  $C_p$  is a finite subgroup of an open torus in  $\mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1$ . The birational map  $f^{-1} : \mathbb{P}_{\bar{k}}^1 \times \mathbb{P}_{\bar{k}}^1 \dashrightarrow \mathbb{F}_n$  preserves this torus. Therefore an MMP-reduction of  $\mathbb{F}_n/C_p$  is a  $k$ -rational minimal toric surface by Lemma 4.2.

**4.3.4. Case 4: Del Pezzo surface of degree 6.** A Del Pezzo surface of degree 6  $X_6$  is isomorphic over algebraically closed field  $\bar{k}$  to blowup  $\mathbb{P}_{\bar{k}}^2$  at three points in general position. It can be assumed that these points have homogenous coordinates  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(0 : 0 : 1)$ .

The automorphism group of  $X_6$  is isomorphic to  $\mathbb{T}^2 \rtimes D_{12}$  where  $\mathbb{T}^2$  is two-dimensional torus over  $\bar{k}$  and  $D_{12}$  acts on the set of  $(-1)$ -curves (the exceptional divisors of the blowup and the proper transforms of lines passing through a pair of points of blowup).

*Remark 4.7.* To work with a Del Pezzo of degree 6 we will use coordinates on  $\mathbb{P}_{\mathbb{k}}^2$ . An equation in these coordinates defines a curve on the open set, which is the Del Pezzo surface of degree 6 without 6  $(-1)$ -curves. At the same time it is clear how the curve intersects each of  $(-1)$ -curves because the Del Pezzo surface of degree 6 is isomorphic to blowup  $\mathbb{P}_{\mathbb{k}}^2$  at three points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ ,  $(0 : 0 : 1)$ .

We have the exact sequence:

$$1 \rightarrow \mathbb{T}^2 \hookrightarrow \mathbb{T}^2 \rtimes D_{12} \twoheadrightarrow D_{12} \rightarrow 1.$$

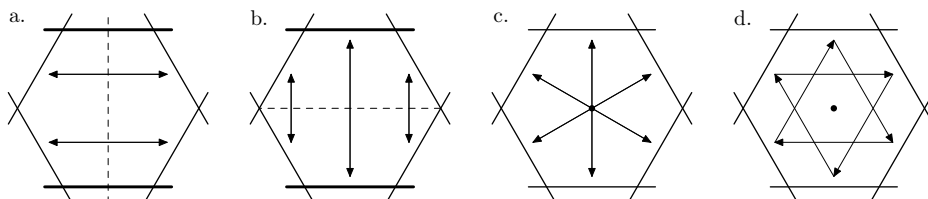
Let  $p$  be a prime and  $C_p$  be a cyclic subgroup of the group  $\text{Aut}(X_6) = \mathbb{T}^2 \rtimes D_{12}$ . The composition of maps

$$C_p \hookrightarrow \mathbb{T}^2 \rtimes D_{12} \twoheadrightarrow D_{12}$$

takes  $C_p$  to identity if  $p > 3$ . Thus one has  $C_p \hookrightarrow \mathbb{T}^2$ . So the group  $C_p$  is a finite subgroup of an open torus in  $X_6$ . An MMP-reduction of  $X_6/C_p$  is a  $\mathbb{k}$ -rational minimal toric surface by Lemma 4.2.

If  $p = 2$  or  $p = 3$  and  $C_p$  is not a subgroup of  $\mathbb{T}^2$  then the group  $C_p$  acts on the set of  $(-1)$ -curves forming a hexagon. There are four nonconjugate cyclic subgroups of prime order in the group  $D_{12}$ . The actions on the hexagon, which sides correspond to  $(-1)$ -curves and vertices correspond to their intersection points, are the following:

- (a) A reflection along a line passing through middles of opposite sides of the hexagon;
- (b) A reflection along a line passing through two opposite vertices of the hexagon;
- (c) The central symmetry;
- (d) The rotation by an angle of  $\frac{\pi}{3}$ .



In the case (a) we have two invariant disjoint  $(-1)$ -curves and the others are not invariant. Therefore the action of  $C_2$  is not minimal and we can contract this pair and get  $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$  considered in the case 2.

In the case (b) we have the invariant pair of disjoint  $(-1)$ -curves (two other pairs of  $(-1)$ -curves are not disjoint). Therefore the action of  $C_2$  is not minimal and we can contract this pair and get  $\mathbb{P}_{\mathbb{k}}^1 \times \mathbb{P}_{\mathbb{k}}^1$  considered in the case 2.

In the case (c) the action of  $C_2$  is conjugate to  $(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ . There are four fixed points:

$$t_1 = (1 : 1 : 1), t_2 = (-1 : 1 : 1), t_3 = (-1 : -1 : 1), t_4 = (1 : -1 : 1).$$

For each pair of these points there is an unique invariant curve with zero selfintersection passing through this pair. Their equations are the following:

$$m_{12} = \{y = z\}, m_{13} = \{x = y\}, m_{14} = \{x = z\},$$

$$m_{23} = \{x = -z\}, m_{24} = \{x = -y\}, m_{34} = \{y = -z\}.$$

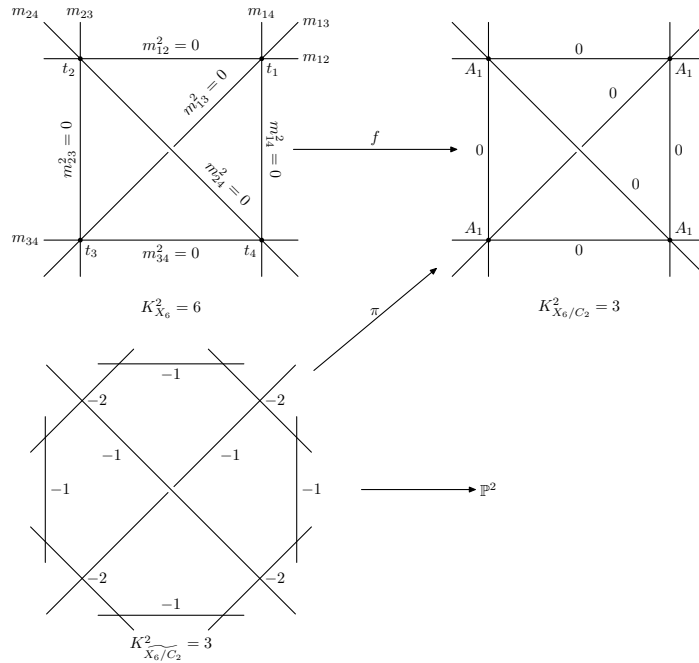
These curves don't intersect at any points on  $X_6$  except  $t_i$ .

Let  $f : X_6 \rightarrow X_6/C_2$  be the quotient map. The four points  $f(t_i)$  are singularities of type  $A_1$  and one has  $(f(m_{ij}))^2 = 0$ ,  $K_{X_6/C_2}^2 = 3$ .

Let  $\pi : \widetilde{X_6/C_2} \rightarrow X_6/C_2$  be the minimal resolution of the singularities. By Remark 4.5 one has:

$$(\pi^{-1}f(t_i))^2 = -2, (\pi^*f(m_{ij}))^2 = -1, (\pi^*f(m_{ij})) \cdot (\pi^*f(m_{kl})) = 0$$

if the pair  $ij$  differs from the pair  $kl$ ,  $K_{\widetilde{X_6/C_2}}^2 = 3$ . Therefore we can equivariantly contract six curves  $\pi^*f(m_{ij})$  and get a surface  $Y$  with  $K_Y^2 = 9$ .  $Y$  is rational so it is isomorphic to  $\mathbb{P}_{\mathbb{k}}^2$ . So  $Y$  is a toric surface and it is  $k$ -rational by Theorem 2.8.



In the case (d) the action of  $C_3$  is conjugate to  $(x : y : z) \mapsto (y : z : x)$ . There are three fixed points:

$$t_1 = (1 : 1 : 1), t_2 = (\omega : \omega^2 : 1), t_3 = (\omega^2 : \omega : 1).$$

For each pair of these points there are exactly two invariant curves with selfintersection one passing through this pair. Their equations are the following:

$$m_{12} = \{\omega x + \omega^2 y + z = 0\}, m_{13} = \{\omega^2 x + \omega y + z = 0\}, m_{23} = \{x + y + z = 0\},$$

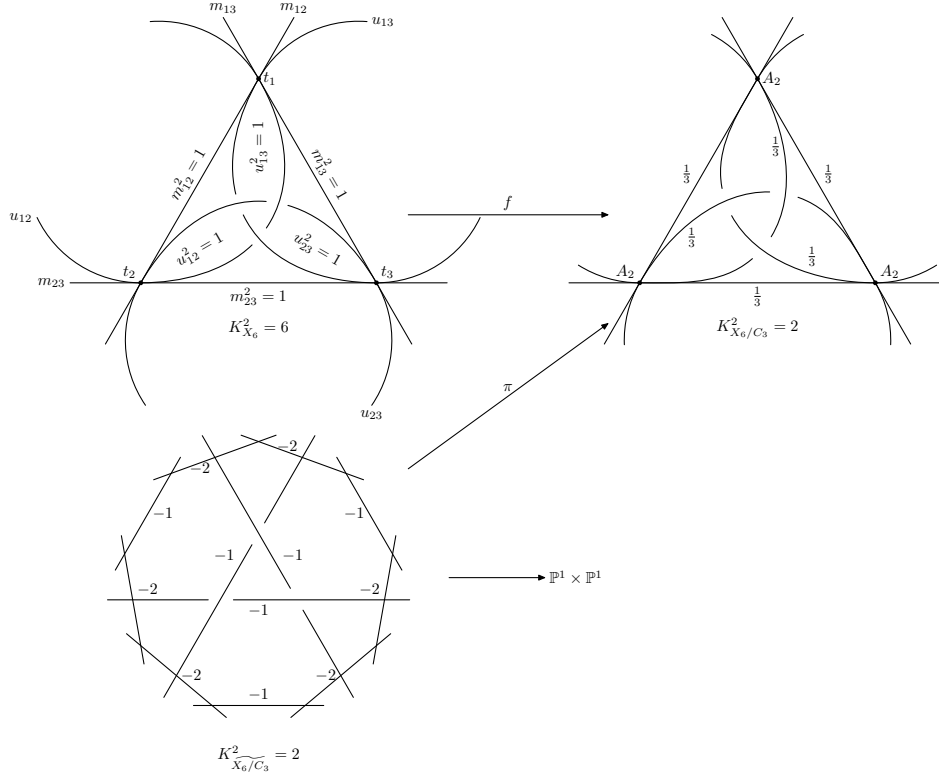
$$u_{12} = \{\omega xy + \omega^2 xz + yz = 0\}, u_{13} = \{\omega^2 xy + \omega xz + yz = 0\},$$

$$u_{23} = \{xy + xz + yz = 0\}.$$

The curves  $m_{ij}$  and  $m_{jk}$  with  $i \neq k$  intersect at the point  $t_j$ , the curves  $m_{ij}$  and  $u_{ij}$  intersect at the points  $t_i$  and  $t_j$ , the curves  $m_{ij}$  and  $u_{jk}$  with  $i \neq k$  intersect with multiplicity 2 at the point  $t_j$ , the curves  $u_{ij}$  and  $u_{jk}$  with  $i \neq k$  intersect at the point  $t_j$ .

Let  $f : X_6 \rightarrow X_6/C_3$  be the quotient map. The three points  $f(t_i)$  are singularities of type  $A_2$  and one has  $(f(m_{ij}))^2 = (f(u_{ij}))^2 = \frac{1}{3}$ ,  $K_{X_6/C_3}^2 = 2$ .

Let  $\pi : \widetilde{X_6/C_3} \rightarrow X_6/C_3$  be the minimal resolution of the singularities. By Remark 4.5 the resolution of each  $A_2$  singularity is a pair of  $(-2)$ -curves intersecting at a point,  $(\pi^* f(m_{ij}))^2 = (\pi^* f(u_{ij}))^2 = -1$ ,  $K_{\widetilde{X_6/C_3}}^2 = 2$ . By direct computation it is easy to check that six curves  $(\pi^* f(m_{ij}))^2$  and  $(\pi^* f(u_{ij}))^2$  are disjoint. Therefore we can equivariantly contract these six curves and get a surface  $Y$  with  $K_Y^2 = 8$  (one can check that  $Y$  is isomorphic to  $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ ). So  $Y$  is a toric surface and it is  $k$ -rational by Theorem 2.8.



**Corollary 4.8.** *Let  $X_6$  be a  $k$ -rational del Pezzo surface of degree 6 and  $G$  be a finite subgroup of  $\Gamma$ -equivariant automorphisms of  $X_6$ . The quotient variety  $X_6/G$  is  $k$ -rational.*

*Proof.* The group  $G$  is a subgroup of  $\mathbb{T}^2 \rtimes D_{12}$ . Therefore there is a normal subgroup  $N = G \cap \mathbb{T}^2$ . An MMP-reduction of  $X_6/N$  is a  $k$ -rational minimal toric surface  $Y$  by Lemma 4.2. The group  $G/N$  is a subgroup of  $D_{12}$ . The center of  $D_{12}$  is  $C_2$ . Let  $M = G/N \cap C_2$  then an MMP-reduction of  $Y/M$  is a  $k$ -rational minimal toric surface  $Z$  by Lemma 4.6 and the group  $(G/N)/M$  is a subgroup of  $\mathfrak{S}_3$ . Let  $L = (G/N)/M \cap C_3$  then an MMP-reduction of  $Z/L$  is a  $k$ -rational minimal toric surface  $W$  by Lemma 4.6 and the group  $((G/N)/M)/L$  is a subgroup of  $C_2$ . An MMP-reduction of  $W/(((G/N)/M)/L)$  is  $k$ -rational toric surface by Lemma 4.6.  $\square$

**4.4. Primitive groups having normal subgroups.** Primitive groups having normal subgroups are groups of type (E), (F), (G) from

Theorem 3.5. Note that  $(E) \subset (F) \subset (G)$ . Moreover they have common subgroup  $N$  generated by  $S = \text{diag}(1, \omega, \omega^2)$ ,  $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$  and  $V^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ .

Note that the group generated by  $\omega I_3$  is a subgroup of all these groups. Therefore for the map  $f : SL_3(\bar{k}) \rightarrow PGL_3(\bar{k})$  one has

$$\text{ord}(f(N)) = 18, \text{ord}(f((E))) = 36, \text{ord}(f((F))) = 72, \text{ord}(f((G))) = 216.$$

Let us show that  $N$  is normal subgroup in all of these groups. We should check up equalities

$$VSV^{-1} = T, VTV^{-1} = V^2SV^2, USU^{-1} = S, UTU^{-1} = S^2T, UV^2U^{-1} = S^2V^2$$

that can be made by direct computation ( $U$  is an element from the classification 3.5). Therefore  $\mathbb{P}_{\bar{k}}^2/G = (\mathbb{P}_{\bar{k}}^2/N)/(G/N)$  where  $G$  is one from the groups (E), (F), (G). The quotients of these groups by the subgroup  $N$  are the following:  $(E)/N = C_2$ ,  $(F)/N = C_2^2$ ,  $(G)/N = \mathfrak{A}_4$ .

Let us consider the quotient variety  $\mathbb{P}_{\bar{k}}^2/N$ . The group  $N$  consists of 18 elements and 9 of them have order 2 and fix different lines. The other elements (excepting identity) have order three and isolated fixed points (three per element). By Hurvitz formula

$$K_{\mathbb{P}_{\bar{k}}^2/N}^2 = \frac{1}{18}(K_{\mathbb{P}_{\bar{k}}^2} - 9l)^2 = 8.$$

Note that fixed points of elements of order 3 don't give us singularities because in their tangent space the group acts as  $\mathfrak{S}_3$ . At the same time for each element of order 2 there is an isolated fixed point. These 9 points are permuted by elements of order 3. So there is one  $A_1$  singular point on the quotient variety  $\mathbb{P}_{\bar{k}}^2/N$ . Its resolution is  $(-2)$ -curve  $C$  and the received surface is isomorphic to  $\mathbb{F}_2$ .

In the case of the action of the group (E) the rationality of the quotient variety  $\mathbb{F}_2/C_2$  directly follows from Lemma 4.2. Note that, in the case of the action of the group (G) on  $\mathbb{P}_{\bar{k}}^2$  the group  $\mathfrak{A}_4$  acts on  $\mathbb{F}_2$ . The action of the group  $\mathfrak{A}_4$  on the base is either trivial, or factors through the homomorphism  $\mathfrak{A}_4 \rightarrow C_3$ , or is a faithful action of  $\mathfrak{A}_4$ . Therefore the action on the base of the normal subgroup  $C_2^2$  of  $\mathfrak{A}_4$ , corresponding to the action of the group (F) on  $\mathbb{P}_{\bar{k}}^2$ , is either trivial or faithful. Let us prove that in the both cases an MMP-reduction of  $\mathbb{F}_2/C_2^2$  is a  $k$ -rational minimal toric surface. The rationality of  $\mathbb{P}_{\bar{k}}^2/(F)$

directly follows from this fact and the rationality of  $\mathbb{P}_{\bar{k}}^2/(G)$  follows from this fact and Lemma 4.2 because  $\mathfrak{A}_4/C_2^2 = C_3$ .

The action of  $C_2^2$  on  $\mathbb{P}_{\bar{k}}^1$  is conjugate to the action of the subgroup in  $PGL_2(\bar{k})$  generated by  $I = \text{diag}(i, -i)$  and  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Each of element  $I, J, K = IJ$  have a pair of fixed points on  $\mathbb{P}_{\bar{k}}^1$ . Let denote these points by  $p_1, p_2, \dots, p_6$ . Note that  $Jp_1 = Kp_1 = p_2$  etc.

If the action of the group  $C_2^2$  is trivial then each fibre of  $\mathbb{F}_2 \rightarrow \mathbb{P}_{\bar{k}}^1$  is invariant. Each fibre intersects with  $(-2)$ -section  $C$  at a fixed point. But the faithful action of the group  $C_2^2$  on fibre which is isomorphic to  $\mathbb{P}_{\bar{k}}^1$  don't have a fixed point. So this case can't be achieved.

If the group  $C_2^2$  acts on the base faithfully let us consider six fibres  $F_i$  over points  $p_i$ . Note that these fibres are permuted by the group  $\mathfrak{A}_4 = (G)/N$ . The action of the element  $I$  on  $F_1$  is either trivial or faithful.

In the first case each  $F_i$  is fixed by one of the elements  $I, J, K$ . Therefore by Hurvitz formula

$$K_{\mathbb{F}_2/C_2^2}^2 = \frac{1}{4}(-2C - 4F - 6F)^2 = 8,$$

where  $F$  is the class of fibre in  $\text{Pic}(\mathbb{F}_2)$ . So  $\mathbb{F}_2/C_2^2$  is a  $k$ -rational toric surface.

If the action of the element  $I$  on  $F_1$  is faithful then there are two  $I$ -fixed points on  $F_1$ . One of them is the intersection of  $F_1$  and  $(-2)$ -section  $C$ , we denote this point by  $p_{-1}$  and the other  $I$ -fixed point on  $F_1$  by  $p_{+1}$ . In the same way we can define points  $p_{\pm 2}, p_{\pm 3}, \dots, p_{\pm 6}$ .

Let  $f : \mathbb{F}_2 \rightarrow \mathbb{F}_2/C_2^2$  be the quotient map. The six points  $f(p_{\pm i})$  are singularities of type  $A_1$  and one has  $(f(F_i))^2 = 0, (f(C))^2 = -\frac{1}{2}, K_{\mathbb{F}_2/C_2^2}^2 = 2$ .

Let  $\pi : \widetilde{\mathbb{F}_2/C_2^2} \rightarrow \mathbb{F}_2/C_2^2$  be the minimal resolution of the singularities. By Remark 4.5 one has

$$(\pi^{-1}f(p_{\pm i}))^2 = -2, (\pi^*f(F_i))^2 = -1,$$

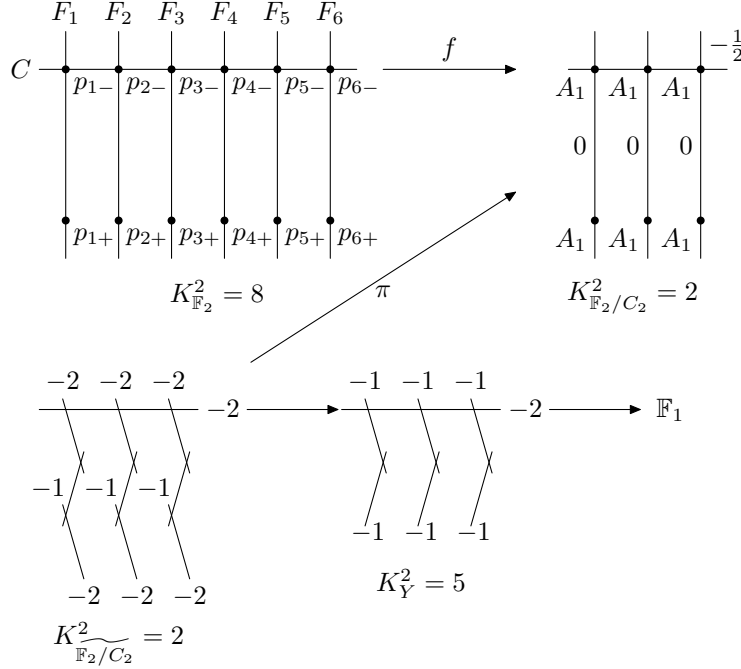
$$(\pi^*f(C))^2 = -2, (\pi^*f(F_i)) \cdot (\pi^*f(C)) = 0.$$

Let  $g : \widetilde{\mathbb{F}_2/C_2^2} \rightarrow Y$  be the contraction of three curves  $\pi^*f(F_i)$ . One has

$$(g\pi^{-1}f(p_{\pm i}))^2 = -1, (g\pi^*f(C))^2 = -2, (g\pi^{-1}f(p_{-i})) \cdot (g\pi^*f(C)) = 1,$$

$$(g\pi^{-1}f(p_{+i})) \cdot (g\pi^*f(C)) = 0, K_Y^2 = 5.$$

Therefore we can equivariantly contract three curves  $g\pi^{-1}f(p_{-i})$  and get a surface  $Z$  with  $K_Z^2 = 8$  (one can check that  $Z$  is isomorphic to  $\mathbb{F}_1$ ). So  $Z$  is a toric surface and it is  $k$ -rational by Theorem 2.8.



**4.5. Simple groups.** Note that each simple group  $G$  is generated by elements of order 2 because its order is even and elements of order 2 generate a normal subgroup. The normal form of an element of order 2 in  $SL_3(\bar{k})$  is  $\text{diag}(-1, -1, 1)$ . This element fixes a line on  $\mathbb{P}_{\bar{k}}^2$  so it is a reflection. It is well-known (a consequence from Shevalley-Shephard-Todd Theorem) that the quotient  $\mathbb{P}_{\bar{k}}^2$  by group generated by reflections is a weighted projective space. It is a toric surface so an MMP-reduction of  $\mathbb{P}_{\bar{k}}^2/G$  is  $k$ -rational by Theorem 2.8.

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